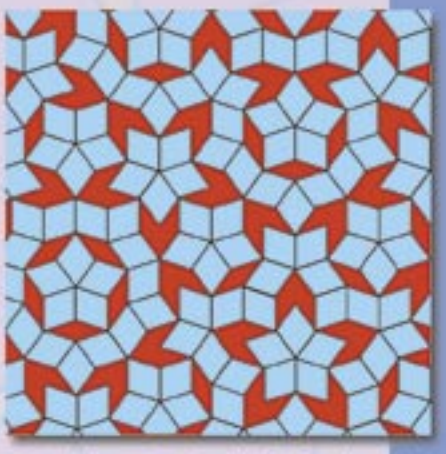
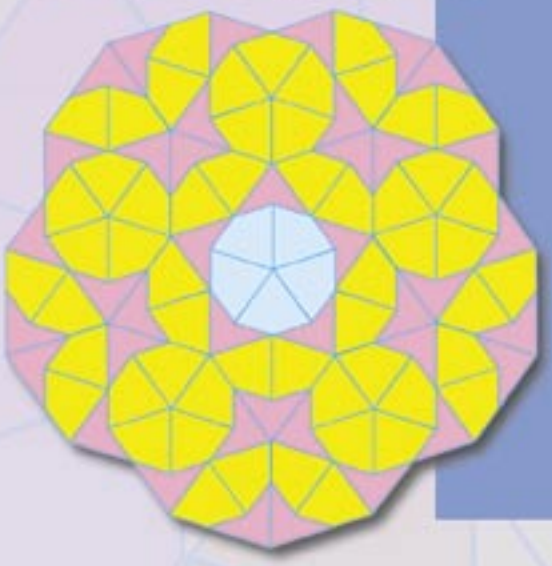




Art and Mathematics Ð Mutual Enrichment

Elizabeth Pumfrey and
Toni Beardon

This article describes some of the work on tessellations by Roger Penrose, an eminent mathematician, and by the artist, M.C.Escher who was inspired by the mathematical influences in the Islamic art he saw in Spain. It requires only a little skill in LOGO programming to draw Penrose tilings and the results are aesthetically very satisfying. For pupils, the challenge to Ôteach the computerÕ to draw Penrose tilings provides progression from the examples cited by Ronnie Goldstien and Dave Pratt and good experience of angle properties of triangles and polygons. Those who are a little more ambitious may like to try drawing Escher-like tessellations in LOGO and to share their ideas in this magazine.

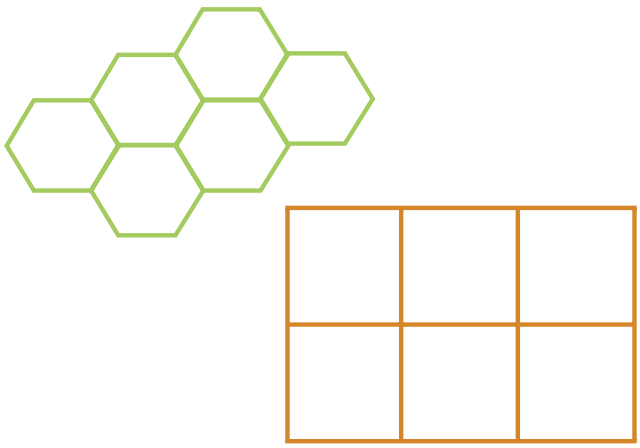
We give links to websites where you can explore the art and the mathematics further and find some activities for the classroom. The NRICH website gives related interactivities on the computer, and projects to do away from the computer. Many of the ideas in this article came from colleagues and friends of the NRICH team, in particular from Jenni Back, the Primary Coordinator, and Alan Beardon and Keith Carne of the Department of Pure Mathematics at Cambridge.

This style of geometrical art was a product of the Islamic civilisation brought to Europe by the Arab conquests in Spain in the thirteenth century. There had been no significant development of mathematical scholarship in Europe since the classical Greek

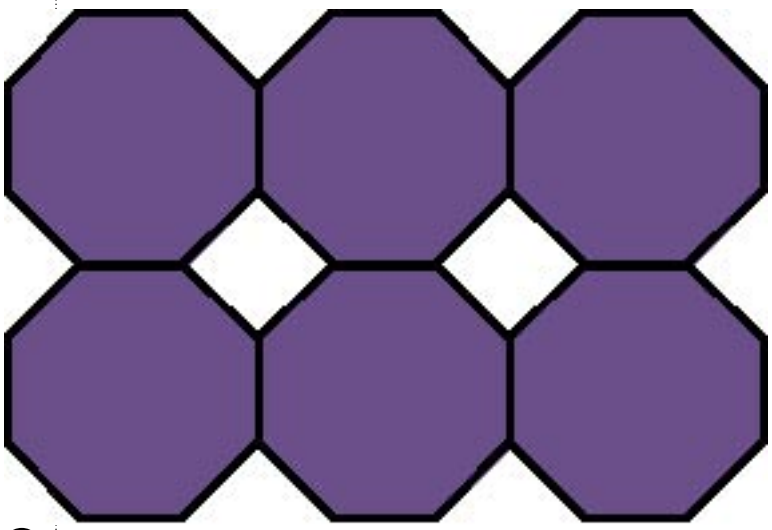
civilisation but throughout this time geometrical knowledge was developed in the Islamic world and expressed in their art. In the ancient Arab universities scholars extended the mathematics known to the Greeks, and there was some exchange of ideas with Chinese and Indian Mathematics.

Tessellation is a system of fitting congruent shapes together to cover a surface, without any gaps or overlapping. The word tessellation itself derives from the Greek tessera, which is associated with four, square and tile. Tessellations are a common feature of decorative art and occur in the natural world all around us.

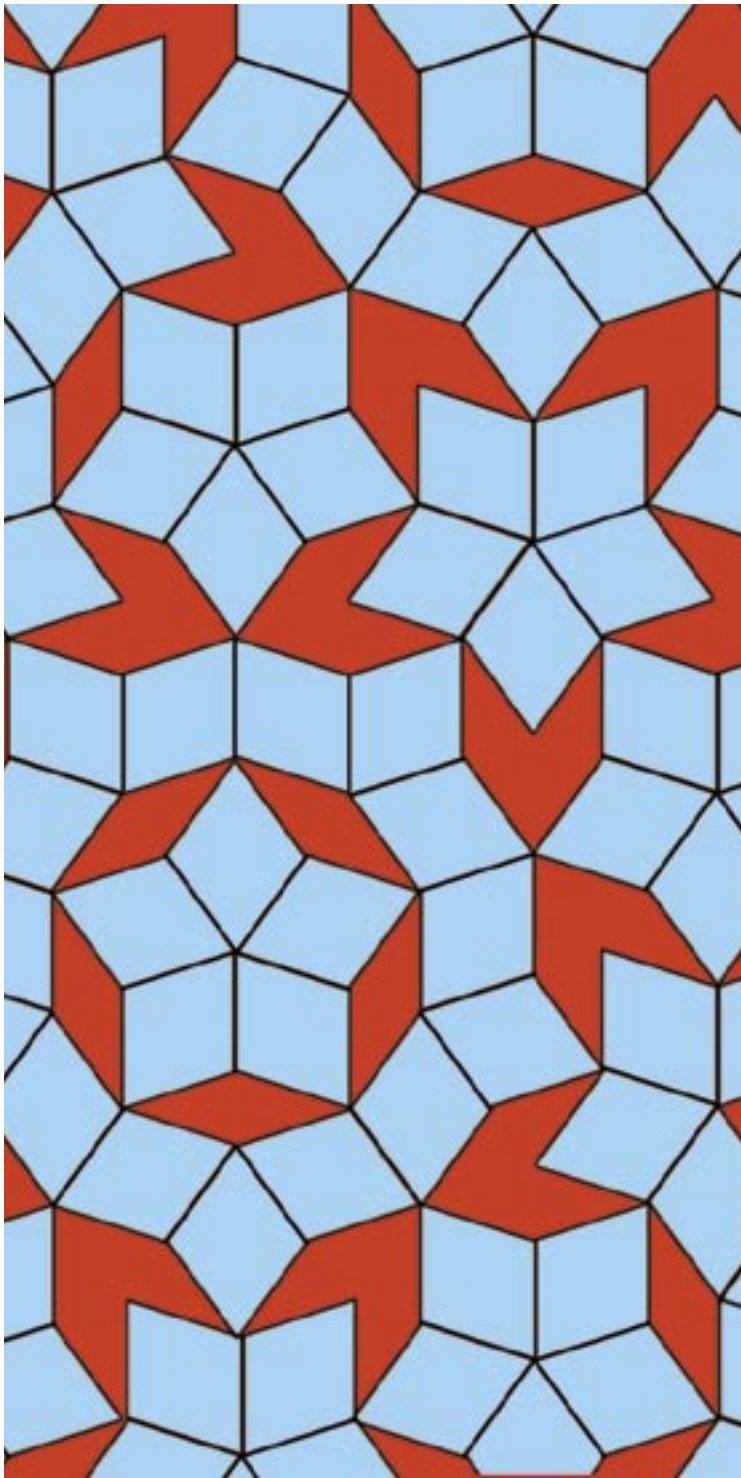
Traditionally, a tessellation pattern is repetitive. The image that we are likely to think of is known as a regular tessellation, where all the shapes are regular and of the same type, for example:
A semi-regular tessellation is made up of two different regular shapes and each vertex (i.e. point where they



meet) is identical. Octagons and squares can be arranged to form a semi-regular pattern:
Roger Penrose took this a step further. While studying for his PhD at Cambridge, Penrose became fascinated by the geometry of covering a plane. He began by investigating combinations of shapes that would produce a repeating unit, but this led on to a search for a pattern with no repetition. This kind of tessellation became known as quasi-periodic, in other words at first glance there appears to be a repeating pattern, but in fact there is not.



Using only pencil and paper, Penrose found such an arrangement but it contained many different shapes. Amazingly, he managed to reduce this to only six, then to just two. The two shapes are both parallelograms and the tessellation is often referred to as ÒKites and DartsÓ:
Although there are small repeated sections, there is no single unit that can be copied to fill the plane. This in itself makes a lovely investigation.

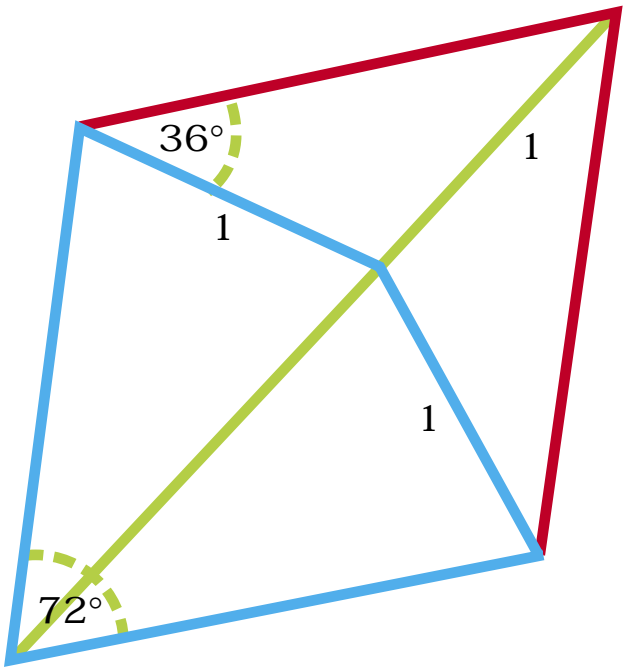


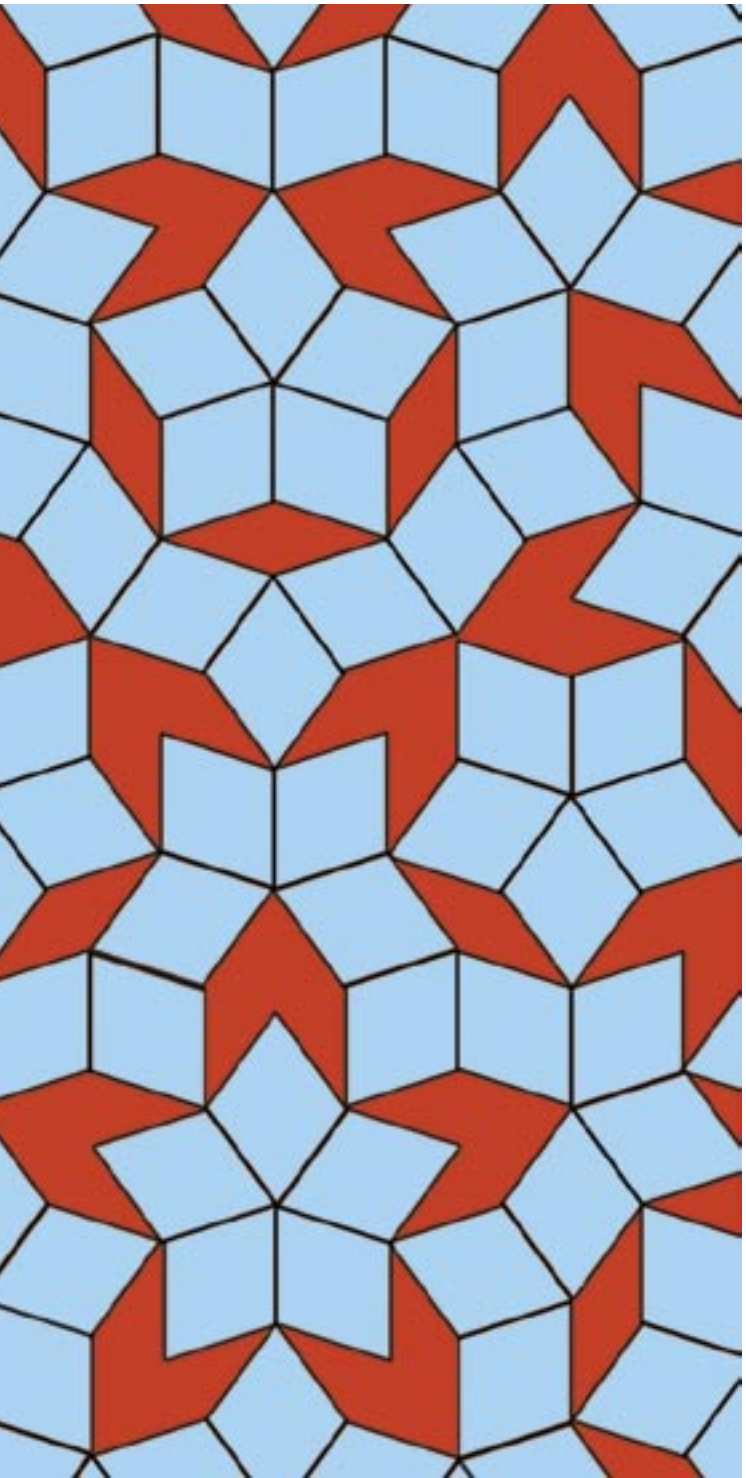
To make a Penrose tessellation for yourself online see the interactivity on the NRICH Prime website at

http://nrich.maths.org/primary/dec01/games.htm

As well as trying this out on your computer, you may like to print the templates of the kite and dart shapes given on the NRICH website so you can make all manner of tessellations yourself. See http://nrich.maths.org/primary/dec01/magazine.htm#making

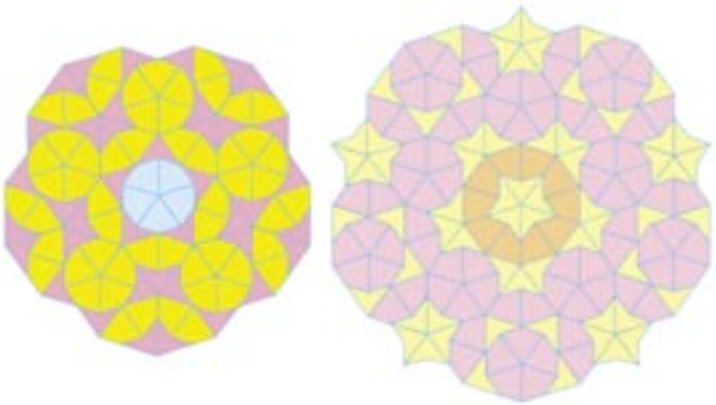
The Penrose tile is a rhombus whose side length is ϕ, phi - the Golden section, equal to $(1 + \sqrt{5}) \div 2$ (approximately 1.618) and whose interior angle is 72 degrees. This tile is split into two quadrilaterals, the dart and the kite and each of these is split symmetrically into two isosceles triangles.





One of the best ways to explore tilings is to draw them for yourself using LOGO. To help you to get started we give a simple program which draws darts and kites and starts to draw the tiling shown below. You can modify and extend these programs to make smaller tiles, to copy the tessellations illustrated here and to make some more of your own design.

```
to dart
fd 100 lt 36
fd 100 rt 144
fd 161.8 rt 108
fd 161.8 rt 144
end
```



Reproduced with permission from I.Alexander
easyweb.easynet.co.uk/~iany/patterns/penrose.htm

```
to fivekites
repeat 5 [kite rt 72]
end

to kite
fd 161.8 rt 108
fd 100 rt 36
fd 100 rt 108
fd 161.8 rt 108
end

to spray
cs fivekites
repeat 5 [rt 36 pu fd 161.8 pd
lt 108 dart rt 36 kite rt 72 kite
pu bk 161.8 rt 36 pd]
end
```

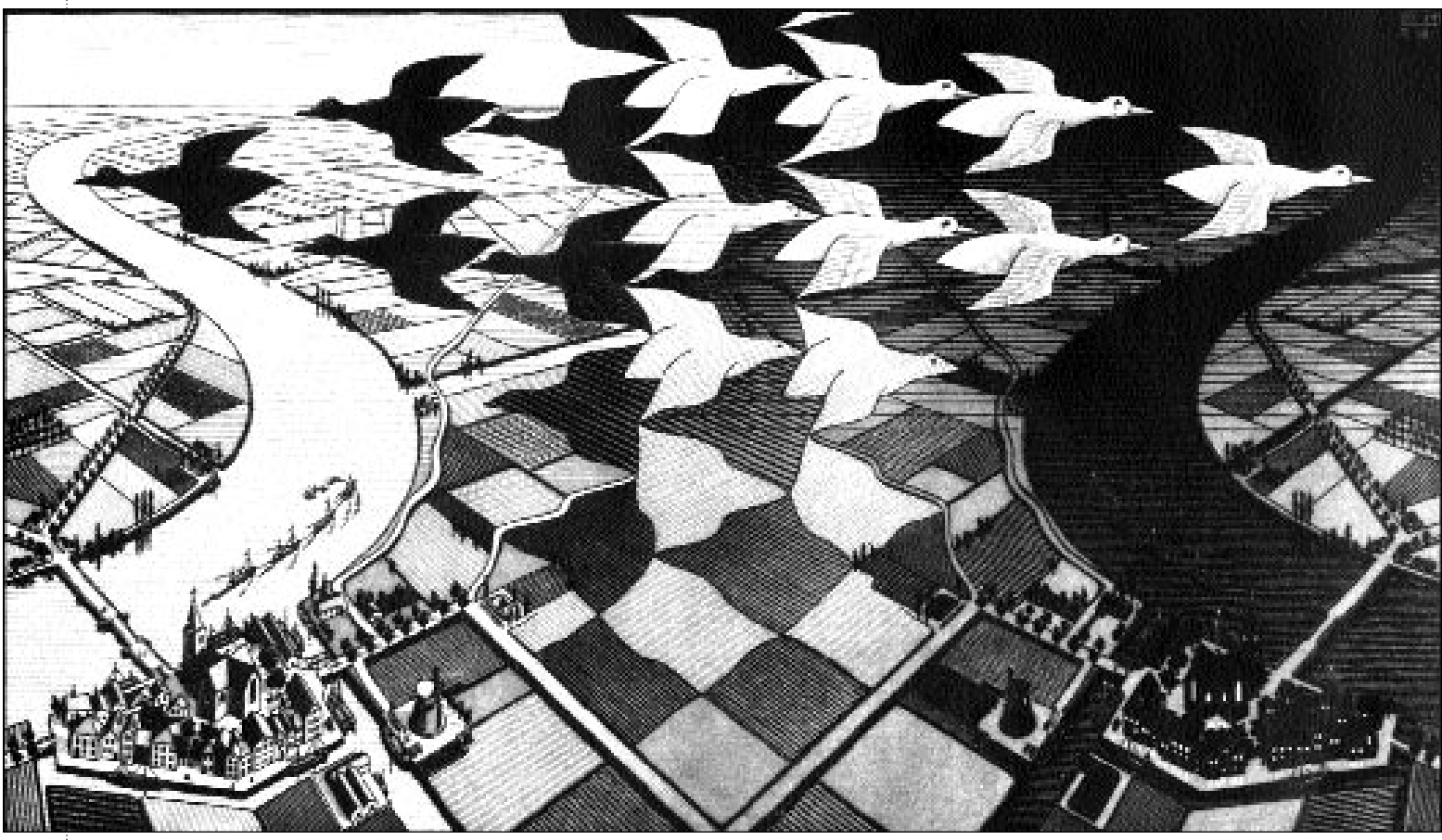
On the NRICH website the sequence of articles by Lyndon Baker (http://nrich.maths.org/mathsf/journalf/rb_logoland.html) takes you from the beginnings of LOGO programming, progresses through an analysis of the seventeen wallpaper patterns, and provides many more tessellations and geometrical patterns for you to draw using LOGO.

Another remarkable man who contributed enormously to the study of tessellation was the Dutch artist M.C.Escher. Born in 1898, initially he concentrated on sketching scenery and surrounding objects. Gradually, EscherÕs work began to change. Rather than drawing what he saw, Escher started to express ideas he had in his mind. He was able to create spatial illusions and detailed repeating patterns.

This picture was completed in 1930 and is called ÒDay and NightÓ. Can you see how Escher has used black and white bird silhouettes to show the change from day to night?

On a visit to Alhambra Escher became fascinated by the Arabic tessellation patterns in the tiling and started to experiment with shapes and mirror images. The Alhambra Palace in Granada is one of the finest examples of Islamic architecture. Due to the fact that Islamic people were forbidden to represent humans and living creatures in any decorative form, they became masters of tiling patterns and tessellations. The intricate detail of the Alhambra Palace is well worth a look and can be seen at:

www.greatbuildings.com/buildings/The_Alhambra.html



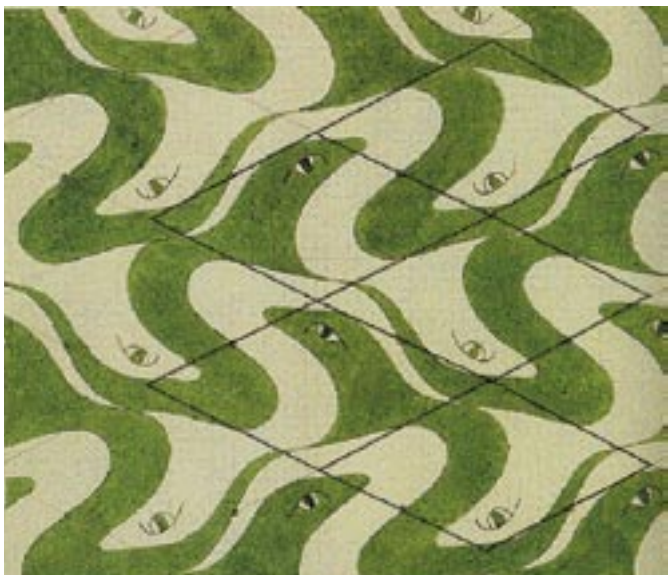
In his earlier pictures like ÔDay and NightÕ the shapes fitting together were not copies of each other but cleverly metamorphosed from one object to another by a sequence of small changes. In 1936, Escher made a second trip to Alhambra and many of his later pictures contained amazing tessellations of varying complexity with shapes, transformed and repeated by reflections, rotations and translations but not otherwise distorted. The mathematics lies in the way the pattern repeats itself, and which geometrical transformations are combined, not in what the shapes depict.

This tessellation consists of interlocking rectangles which contain the fish and boat images. The rectangles are translated diagonally to produce the entire picture. This is one of the simplest types of tessellation. There are no reflections and no rotations in it.

This second tessellation is much more complicated. It contains interlocking rhombi which have undergone several transformations including translations, rotations and reflections.

Escher had powers of visualisation that gave him a superior intuitive understanding of geometry and he

brought geometry to life in his art. He also discussed mathematics with one of the foremost geometers in the world, namely H.S.M.Coxeter in Toronto. Escher gained a great deal of respect from mathematicians for his work and he lectured on art, mathematics and science.



EscherÕs drawings began to show a new theme towards the late 1950s. He perfected a technique known as Òshowing infinityÓ which gave the impression that his sketches were endless. He did this by making the objects gradually smaller towards the edge of the paper. Have a look!

Escher used the three different geometries of the sphere, the flat plane, and the hyperbolic plane to give wonderful pictures. These are from The Magic of M.C. Escher, Thames and Hudson, 2000 (ISBN 0-500-97591-4). For more about these strange geometries and some projects you might like to try, see the NRICH website at:

http://nrich.maths.org/MOTIVATE/conf8/index.html

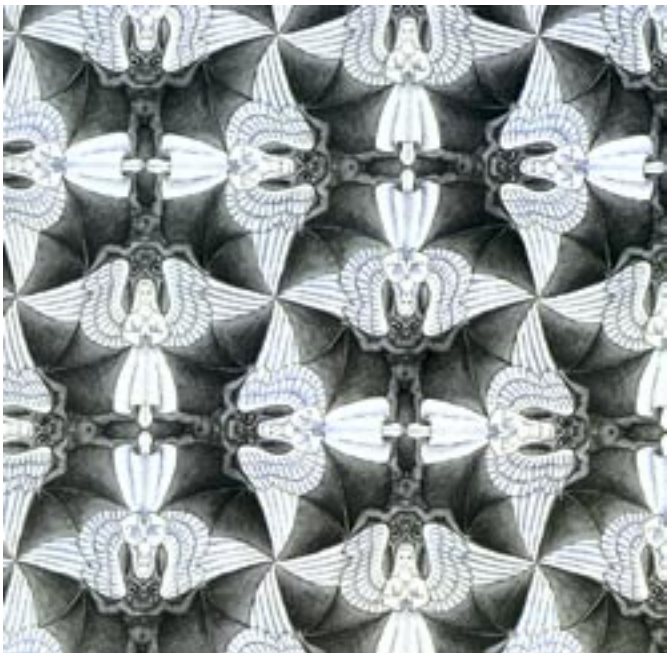
You can find out more about Escher and his work from these websites:

- www.worldofescher.com
- www.domaindlx.com/Escher
- library.thinkquest.org/11750/?tqskip=1
- users.erols.com/ziring/escher.htm
- www.mcescher.com

Closer to home, the Great Court of the British Museum in London is another magnificent example of architectural tessellation. Or is it?

Its incredible roof is entirely made up of triangular shaped panes of glass supported by a steel framework.

Sphere with Angels and Devils



Regular Division of the Plane # 45

Circle limit IV 1960 (Heaven and Hell)

However, although the panes have been fitted together without overlapping or gaps, this is not a true tessellation.

In order to create the curved 3-dimensional roof, the pieces of glass cannot be identical.

Tessellations lend themselves very well to exploration. When a surface is covered tessellating designs are often seen, both natural and man-made. Both regular and irregular 3D tessellations can frequently be found in nature. What about scales on a fish or a tortoise shell? A pineappleÕs ÒskinÓ, ears of corn and honeycomb could all be classed as tessellations in three dimensions.
For further information see the following websites:

- mathartfun.com
- worthhall.demon.co.uk/theory/lister/tessel.htm
- www.easyweb.easynet.co.uk/~iany/patterns/tessellations_welcome.htm
- planetpatchwork.com/tesselat.htm
- worldofescher.com/misc/penrose.html

Elizabeth Pumfrey and Toni Beardon, NRICH Project, University of Cambridge.

The work that went into the research, production and preparation of this document has to be supported somehow.

ATM receives its financing from only two principle sources: membership subscriptions and sales of books, software and other resources.

Membership of the ATM will help you through

Now, this bit is important - you must read this

- Six issues per year of a professional journal, which focus on the learning and teaching of maths. Ideas for the classroom, personal experiences and shared thoughts about developing learners' understanding.
- Professional development courses tailored to your needs. Agree the content with us and we do the rest.
- Easter conference, which brings together teachers interested in learning and teaching mathematics, with excellent speakers and workshops and seminars led by experienced facilitators.
- Regular e-newsletters keeping you up to date with developments in the learning and teaching of mathematics.
- Generous discounts on a wide range of publications and software.
- A network of mathematics educators around the United Kingdom to share good practice or ask advice.
- Active campaigning. The ATM campaigns at all levels towards: encouraging increased understanding and enjoyment of mathematics; encouraging increased understanding of how people learn mathematics; encouraging the sharing and evaluation of teaching and learning strategies and practices; promoting the exploration of new ideas and possibilities and initiating and contributing to discussion of and developments in mathematics education at all levels.
- Representation on national bodies helping to formulate policy in mathematics education.
- Software demonstrations by arrangement.

Personal members get the following additional benefits:

- Access to a members only part of the popular ATM website giving you access to sample materials and up to date information.
- Advice on resources, curriculum development and current research relating to mathematics education.
- Optional membership of a working group being inspired by working with other colleagues on a specific project.
- Special rates at the annual conference
- Information about current legislation relating to your job.
- Tax deductible personal subscription, making it even better value

Additional benefits

The ATM is constantly looking to improve the benefits for members. Please visit www.atm.org.uk regularly for new details.

LINK: www.atm.org.uk/join/index.html